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Perfect Spline Solutions to L^∞ Extremal Problems*

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INTRODUCTION

Recently, S. Karlin [6] announced some fundamental results about perfect spline functions and their extremal properties and related results concerning the estimation of best constants. One of these results concerns the minimization problem

$$\| s^{(n)} \|_{L^\infty(0,1)} = \min\{\| f^{(n)} \|_{L^\infty(0,1)} : f \in U \subset W^{n,\infty}(0, 1)\} \tag{1}$$

where the flat U in the Sobolev space $W^{n,\infty}(0, 1)$, $n \geqslant 1$, is defined by prescribed interpolation of values $r_1 ,..., r_{n+k}$ on a mesh $0 = x_1 \leqslant \cdots \leqslant x_{n+k} = 1$ which permits at most n coincident values of the mesh points. Interpolation of derivatives through order $\nu - 1$ is understood at a mesh point of multiplicity ν. A basic result announced by Karlin is that the minimization problem (1) admits a perfect spline solution of the form

$$s(x) = c\left[x^n + 2 \sum_{i=1}^{k-1} (-1)^i (x - \xi_i)_+^n\right] + \sum_{\nu=0}^{n-1} a_\nu x^\nu \tag{2}$$

where $c, a_0 ,..., a_{n-1}$ are real constants and $0 < \xi_1 < \xi_2 < \cdots < \xi_{k-1} < 1$ are the knots of s.

We shall show in this paper that perfect spline solutions can be obtained for a strictly wider class of constrained minimization problems than those considered by Karlin. Our result admits certain Hermite-Birkhoff interpolation constraints, viz., those that are locally poised in a sense to be made precise later. The result is expressed in Theorem 3 of Section 1. The advantage of our methods is their simplicity. Although the analysis is lengthy and delicate, it is accessible via the calculus. The techniques make fundamental use of the existence, demonstrated in [4], of *piecewise* perfect spline solutions

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to constrained L^∞ minimization problems as well as the existence of a fundamental core interval of uniqueness. These results, which were obtained by basic convexity and functional analysis techniques, are reviewed as Theorems 1 and 2 in Section 1. We state them in somewhat more general form than is actually needed.

Our result does not exactly reduce to Karlin's in the case of interpolation of successive derivatives at nodes. Specifically, Karlin has a precise global estimate of $k - 1$ for the maximum number of knots on $(0, 1)$, i.e., $n + 1$ less than the number of interpolation constraints. In contrast to the global approach taken by Karlin, ours is essentially a local one. We show that between any two distinct nodes x_i and x_{i+1} our perfect spline solution has at most n knots ($n - 1$ if on the core interval of uniqueness). Our method of proof is also local. We show that there exists an arbitrarily small perturbation of the data such that the uniquely determined solution of the perturbed problem on the core interval can be extended to a perfect spline solution of the perturbed problem. We then establish appropriate convergence as the perturbations tend to zero.

We close the introduction with a brief historical account of related results. The emergence of perfect splines as extremals of L^∞ variational problems seems to date from the Achieser–Favard–Krein theorem [1, 2] on the best L^∞ approximation of periodic functions by trigonometric polynomials. Favard [3] later asserted the importance of perfect spline solutions for the problem (1). Glaeser [5] gave the first concrete solution of (1) in the special case of two nodes, each of multiplicity n. He demonstrated the existence of a unique perfect spline solution with at most $n - 1$ interior knots. Louboutin [7] displayed a closed form solution of the problem considered by Glaeser under very special choices of the interpolation. We refer the reader to the informative related paper of Schoenberg [8] for an account of this and related results. Smith [9], in his dissertation, proved the existence of a piecewise perfect spline solution of problem (1) with simple nodes. Finally, [4] established the existence of a fundamental core interval of uniqueness and considerably extended the range of applicable extremal problems.

1. Piecewise-Perfect and Perfect Spline Solutions

Let m points $x_1 < x_2 < \cdots < x_m$ be specified in $\mathbb{R}$ together with a positive integer n. Associated with each of these points x_i, we consider the linear functionals L_{ij} defined by

$$L_{ij} = \sum_{\nu=0}^{n-1} a_{ij}^{(\nu)} D^\nu(\cdot)(x_i), \qquad j = 1,\dots, k_i, \qquad i = 1,\dots, m,$$

for prescribed real numbers $a_{ij}^{(\nu)}$ such that, for each i, the k_i n-tuples $(a_{ij}^{(0)},\ldots, a_{ij}^{n-1})$ are linearly independent. Here $1 \leqslant k_i \leqslant n$ for $i = 1,\ldots, m$, and the L_{ij} are taken to operate on the real Sobolev class

$$W^{n,\infty}(x_1, x_m) = \{f \in C^{n-1}[x_1, x_m] : f^{(n-1)} \text{ is absolutely continuous}, f^{(n)} \in L^\infty(x_1, x_m)\}.$$

Let L be a nonsingular linear differential operator on $[x_1, x_m]$ of order n of the form

$$L = D^n + \sum_{j=0}^{n-1} c_j D^j,$$

where $c_j \in C[a, b], j = 0, 1,\ldots, n - 1$. We consider the constrained minimization problem over $W^{n,\infty}(x_1, x_m)$:

$$\| Ls \|_{L^\infty(x_1,x_m)} = \alpha = \inf\{\| Lf \|_{L^\infty(x_1,x_m)} : f \in U\} \tag{1.1}$$

$$U = \{f \in W^{n,\infty}(x_1, x_m): L_{ij}f = r_{ij}, 1 \leqslant j \leqslant k_i, 1 \leqslant i \leqslant m\}, \tag{1.2}$$

for prescribed real numbers r_{ij}.

THEOREM 1. *The minimization problem* (1.1) *has a solution* $s \in W^{n,\infty}(x_1, x_m)$ *and the class* $S(U)$ *of all such solutions* s *for a fixed choice of* U *is a convex set. Let* $S_1(U) = S(U)$ *and, for* $2 \leqslant i \leqslant m$, *let* $S_i(U)$ *consist of all solutions to the minimization problem*

$$\alpha_{i-1} = \inf\{\| Ls \|_{L^\infty(x_{i-1},x_i)} : s \in S_{i-1}(U)\}.$$

Then each $S_i(U)$ *is nonempty; in particular, there is an* S_* *in*

$$S_m(U) = \bigcap_{i=1}^{m} S_i(U).$$

In order to obtain the existence of piecewise perfect spline solutions to (1.1) as well as the existence of a core interval of uniqueness we must make additional assumptions regarding the differential operator L and the linear functionals L_{ij}. Regarding L we assume further:

(I) $c_j \in C^j[a, b]$; the null space of the formal adjoint L^* of L given by

$$L^*f = (-1)^n D^n f + \sum_{j=0}^{n-1} (-1)^j D^j(c_j f)$$

is spanned by a Tchebycheff system, i.e., if $u \in C^n[x_1, x_m]$ satisfies $L^*u = 0$ on $[x_1, x_m]$ and if $u(y_1) = \cdots = u(y_n) = 0$ for any set of n points

$$x_1 \leqslant y_1 < \cdots < y_n \leqslant x_m$$

then $u = 0$ on $[x_1, x_m]$.

In order to state conveniently our hypothesis on the L_{ij} we define the integer n_0 to be the maximum positive integer satisfying the following property: for any n_0 consecutive points among $x_1, \ldots, x_m$ the sum of the integers k_i associated with these points does not exceed n. Clearly, we have $1 \leqslant n_0 \leqslant n$. Then our assumption about the L_{ij} is as follows:

(II) (a) For every n_0 consecutive points $x_{\lambda_0}, \ldots, x_{\lambda_0+n_0-1}$ and prescribed values y_{ij} there is a function u in the null space of L satisfying $L_{ij}u = y_{ij}$, $j = 1, \ldots, k_i$, $i = \lambda_0, \ldots, \lambda_0 + n_0 - 1$.

(b) For every $n_0 + 1$ consecutive points $x_{\lambda_0}, \ldots, x_{\lambda_0+n_0}$ such that

$$\sum_{\nu=\lambda_0}^{\lambda_0+n_0} k_\nu \geqslant n + 1$$

the equations

$$L_{ij}u = 0, j = 1, \ldots, k_i, i = \lambda_0, \ldots, \lambda_0 + n_0$$

for u in the null space of L imply $u \equiv 0$.

THEOREM 2. *Suppose* (I) *and* (II) *are satisfied. Then there is a core interval* $J = [x_{\lambda_1}, x_{\lambda_2+n_0}]$ *for some* $1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant m - n_0$ *satisfying*

$$\sum_{i=\lambda_1}^{\lambda_2+n_0} k_i \geqslant n + 1$$

such that any two solutions of (1.1) *agree on* J. *Moreover, if* $s \in S(U)$ *then* $|Ls| = \alpha$ a.e. *on J. If* s_* *is chosen as in Theorem* 1, *then* s_* *is unique in* $S_m(U)$. *Moreover,* s_* *satisfies the property that* $|Ls_*|$ *is equivalent to a step function on* (x_1, x_m) *with discontinuities restricted to* $x_2, \ldots, x_{m-1}$ *and, on* (x_i, x_{i+1}), $i = 1, \ldots, m - 1$, Ls_* *is equivalent to a step function with at most* $n - 1$ *discontinuities on each such interval.*

When $L = D^n$, Theorem 2 asserts the existence of a piecewise perfect spline solution s_* to (1.1), i.e., s_* is a perfect spline on each (x_i, x_{i+1}) with $|s^{(n)}| = \alpha_i \leqslant \alpha$ and s_* possessing at most n knots on $[x_i, x_{i+1})$, $i = 1, \ldots, m - 1$. The hypothesis (I) is automatically satisfied for the operator $L = D^n$ and (II) is satisfied, e.g., if the L_{ij} are given by

$$L_{ij}f = D^j f(x_i), \quad j = 0, 1, \ldots, k_i - 1, \quad i = 1, \ldots, m.$$

We are now prepared to state our result on the existence of perfect spline solutions to the extremal problem

$$\| D^n s \|_{L^\infty(x_1, x_m)} = \alpha = \inf\{\| D^n f \|_{L^\infty(x_1, x_m)} : f \in U\} \tag{1.3}$$

where U is given by (1.2).

THEOREM 3. *There is a perfect spline solution s to the extremal problem* (1.3), *provided the functionals L_{ij} satisfy hypothesis* (II). *s has the property that $D^n s = \pm\alpha$ except at a finite number of points of discontinuity of $D^n s$, which cannot exceed n in number on (x_i, x_{i+1}) for each $i = 1, \dots, m - 1$.*

2. PERFECT SPLINE EXTREMALS

In this section we give a proof of Theorem 3. The proof is aided by two propositions, the first of which is a perturbation result.

DEFINITION. A spline s of degree n on $[\alpha, \beta]$ is said to have k knots $\alpha < \xi_1 < \cdots < \xi_k < \beta$ on (α, β) and $k + 1$ knots $\alpha, \xi_1, \dots, \xi_k$ on $[\alpha, \beta)$ if the representation

$$s(x) = P(x) + \lambda_0 \frac{(x - \alpha)^n}{n!} + \sum_{j=1}^{k} 2\lambda_j \frac{(x - \xi_j)_+^n}{n!} \tag{2.1}$$

holds for s on $[\alpha, \beta]$ for P a polynomial of degree $n - 1$ and real numbers $\lambda_0, \dots, \lambda_k$. s is a perfect spline if $\lambda_0 = -\lambda_1 = \lambda_2 = \cdots = (-1)^k \lambda_k \neq 0$.

PROPOSITION 1. *Let $(s_0, \dots, s_{n-1})$ and $(S_0, \dots, S_{n-1})$ be arbitrary n-tuples of real numbers. For each $\epsilon > 0$ there is an n-tuple $(r_0, \dots, r_{n-1})$ satisfying*

$$| s_\nu - r_\nu | < \epsilon, 0 \leqslant \nu \leqslant n - 1,$$

such that the equality

$$D^\nu t(1) = r_\nu, \qquad 0 \leqslant \nu \leqslant n - 1$$

fails for every perfect spline t on $[0, 1]$ *with at most $n - 1$ knots on* $[0, 1)$ *for which $D^\nu t(0) = S_\nu$, $0 \leqslant \nu \leqslant n - 1$.*

Proof. Any perfect spline s on $[0, 1]$ with at most $n - 1$ knots on $[0, 1)$ such that $D^\nu s(0) = S_\nu$, $0 \leqslant \nu \leqslant n - 1$, is of the form

$$s(x) = P(x) + \frac{\lambda}{n!} x^n + 2\lambda \sum_{j=1}^{n-2} (-1)^j (x - a_j)_+^n / n!$$

with $-\infty < \lambda < \infty$ and $0 \leqslant a_1 \leqslant \cdots \leqslant a_{n-2} < 1$ for a fixed P of degree $n-1$. Now let F be the function defined on the set,

$$D = \{(\mu, \xi_1, ..., \xi_{n-2}) : -\infty < \mu < \infty, 0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_{n-2} \leqslant 1\},$$

with image in $\mathbb{R}^n$, given by

$$F(\mu, \xi_1, ..., \xi_{n-2}) = \left\{D^\nu P(1) + \frac{\mu}{(n-\nu)!} + 2\mu \sum_{j=1}^{n-2} (-1)^j (1-\xi_j)^{n-\nu}/(n-\nu)!\right\}_{\nu=0}^{n-1}.$$

The function F is in $C^\infty(\bar{D})$ and hence its image has no interior in $\mathbb{R}^n$. In particular, there are points, arbitrarily close to any point in $\mathbb{R}^n$, which are not in the set $F(D)$; specifically, there are points arbitrarily close to $\{s_\nu\}_{\nu=0}^{n-1}$ which are not in the image of $F(D)$, which proves the proposition.

Our next proposition is the core result in the proof of Theorem 3.

PROPOSITION 2. *Let s be a spline of degree n on $[-1, 1]$ with $D^n s = \pm\lambda$ on $(-1, 0)$ and $D^n s = \pm\beta$ on $(0, 1)$ where $|\beta| \leqslant |\lambda|$. Suppose further that there is not a perfect spline t on $[0, 1]$ with at most $n-1$ knots on $[0, 1)$ satisfying*

$$D^\nu t(0) = D^\nu s(0)$$
$$D^\nu t(1) = D^\nu s(1) \qquad \text{for} \quad \nu = 0, ..., n-1.$$

Then there is a perfect spline S on $[-1, 1]$ with $S = s$ on $[-1, 0]$ and $S^{(\nu)}(1) = s^\nu(1)$ for $\nu = 0, ..., n-1$. Further, S can be chosen so that it has at most n knots on $(0, 1)$.

Proof. Let t be the unique solution of the extremal problem

$$\begin{aligned} 0 < \tilde{\beta} &= \| t^{(n)} \|_{L^\infty(0,1)} \\ &= \inf\{\| D^n f \|_{L^\infty(0,1)} : f \in W^{n,\infty}(0, 1), D^\nu f(0) = D^\nu s(0) \\ &\qquad \text{and } D^\nu f(1) = D^\nu s(1), 0 \leqslant \nu \leqslant n-1\}. \end{aligned} \tag{2.2}$$

Then $\tilde{\beta} \leqslant |\beta|$ and there is no loss in assuming $|\beta| = \tilde{\beta}$. By Theorem 2, t is a perfect spline with at most $n-1$ knots on $(0, 1)$. Let ℓ be the function in $W^{n,\infty}(-1, 1)$ which agrees with s on $[-1, 0]$ and with t on $[0, 1]$. We choose the sign of λ so that $\ell^{(n)} = \lambda$ holds near and to the left of zero. Then, for $0 \leqslant x \leqslant 1$, by the hypotheses of the proposition,

$$\ell(x) = p(x) + \frac{\beta' - \lambda}{n!} x^n + \sum_{j=1}^{n-1} (-1)^j \frac{2\beta'}{n!} (x - b_j)_+^n$$

where $0 < b_1 < \cdots < b_{n-1} < 1$, p is a polynomial of degree n satisfying

$$\begin{cases} p^{(\nu)}(0) = s^{(\nu)}(0), & 0 \leqslant \nu \leqslant n-1 \\ p^{(n)}(0) = \lambda \end{cases}$$

and $\beta' = \pm\beta$. If $|\beta| = |\lambda|$ then ℓ provides an appropriate extension. Hence, suppose $0 < |\beta| < |\lambda|$. We wish to find numbers

$$0 < c_1 < \cdots < c_n < 1$$

satisfying

$$2\lambda \sum_{j=1}^{n} (-1)^j (1 - c_j)^{n-\nu} = \beta' - \lambda + 2\beta' \sum_{j=1}^{n-1} (-1)^j (1 - b_j)^{n-\nu},$$
$$0 \leqslant \nu \leqslant n-1, \quad \text{(2.3 i)}$$

if β'/λ is positive and similar numbers satisfying

$$-2\lambda \sum_{j=1}^{n} (-1)^j (1 - c_j)^{n-\nu} = \beta' + \lambda + 2\beta' \sum_{j=1}^{n-1} (-1)^j (1 - b_j)^{n-\nu},$$
$$0 \leqslant \nu \leqslant n-1 \quad \text{(2.3 ii)}$$

if β'/λ is negative. Indeed, Eq. (2.3 i) express the interpolation conditions

$$s_*^{(\nu)}(1) = \ell^{(\nu)}(1), \qquad 0 \leqslant \nu \leqslant n-1,$$

for the perfect spline

$$s_*(x) = p(x) + 2\lambda \sum_{j=1}^{n} (-1)^j \frac{(x - c_j)_+^n}{n!} \qquad \text{(2.4 i)}$$

on [0, 1] whereas Eqs. (2.3 ii) express the conditions

$$S_*^{(\nu)}(1) = \ell^{(\nu)}(1), \qquad 0 \leqslant \nu \leqslant n-1,$$

for the perfect spline

$$S_*(x) = p(x) - 2\lambda \frac{x^n}{n!} - 2\lambda \sum_{j=1}^{n} (-1)^j \frac{(x - c_j)_+^n}{n!} \qquad \text{(2.4 ii)}$$

on [0, 1]. We show that (2.3 i) and (2.3 ii) can be reduced to the same system. Divide (2.3 i) by -2λ and (2.3 ii) by 2λ, set $x_j = 1 - c_j$, $j = 1,\ldots, n$ and set $B_j = 1 - b_j$, $1 \leqslant j \leqslant n-1$. Then both systems reduce to the system, in the unknowns $1 > x_1 > \cdots > x_n > 0$,

$$\sum_{j=1}^{n} (-1)^{j+1} x_j^k = \tfrac{1}{2}(1 - \rho) + \rho \sum_{j=1}^{n-1} (-1)^{j+1} B_j^k, \qquad 1 \leqslant k \leqslant n, \qquad (2.5)$$

where ρ is a fixed number, $0 < \rho < 1$, and $1 > B_1 > \cdots > B_{n-1} > 0$.

Now let

$$D = \{x = (x_1, \dots, x_n) : 1 > x_1 > \cdots > x_n > 0\}$$

and let $F : \bar{D} \to \mathbb{R}^n$ be defined by

$$F(x_1, \dots, x_n) = \left\{ \sum_{j=1}^{n} (-1)^{j+1} x_j^k \right\}_{k=1}^{n}. \tag{2.6}$$

Note that $F \in C^\infty(\bar{D})$ and that F is a local homeomorphism on D since its Jacobian matrix is nonsingular in D. Now let L be the line segment in $\mathbb{R}^n$ joining the points $B = F(B_1, B_2, \dots, B_{n-1}, 0)$ and $A = (1/2, \dots, 1/2)$. We shall show that $L \subset \overline{F(D)}$; specifically we shall show that the interior of L lies in the interior of $\overline{F(D)}$, which is just $F(D)$. This will yield a solution to (2.5) and hence a proof of the proposition via (2.3) and (2.4). Now our hypothesis that there is no perfect spline with fewer than n knots with the desired derivatives at 0 and 1 together with the properties of F, imply that, if L meets the boundary of $F(D)$ at a point O, then either $O = F(1, x_2, \dots, x_n)$ where $1 > x_2 > \cdots > x_n > 0$ or $O = F(x_1, \dots, x_{n-1}, 0)$ where

$$1 > x_1 > \cdots > x_{n-1} > 0.$$

Let $M_0 = \{(x_1, \dots, x_{n-1}, 0) : 1 > x_1 > \cdots > x_{n-1} > 0\}$, let

$$M_1 = \{(1, x_2, \dots, x_n) : 1 > x_2 > \cdots > x_n > 0\}$$

and let N_0 and N_1 be the images under F of M_0 and M_1, respectively. We now show by convexity arguments that L begins in $\overline{F(D)}$ at B and cannot emerge from $F(D)$ until A, if at all. Specifically, we show that, if O is a point of N_0 or N_1 and H is the tangent hyperplane to N_0 (respectively, N_1) at O, then there is a neighborhood of O in which $F(D) \cup N_0 \cup N_1$ lies strictly on one side of H. Then we show that, if L meets N_0 or N_1 at a point O, the line segment L', joining O to A, lies strictly on the same side of H as $F(D)$. It follows that L cannot be outside of $F(D)$ except at A or B.

We prove the two assertions for $O \in N_0$ in detail, sketching the modifications for $O \in N_1$. Let $O = F(\xi_1, \dots, \xi_{n-1}, 0) \in N_0$. Then the tangent vectors to the surface N_0 at O are $(1, 2\xi_j, \dots, n\xi_j^{n-1})$ for $j = 1, \dots, n-1$ and hence a normal vector $(z_1, \dots, z_n)$ to N_0 at O must satisfy

$$\sum_{k=1}^{n} z_k k \xi_j^{k-1} = 0, \qquad j = 1, \dots, n-1.$$

Thus, for $1 \leqslant k \leqslant n$, z_k can be taken to be the coefficient of x^k in the polynomial Q of degree n which is determined by the conditions $Q(0) = 0$ and

$$Q'(x) = \prod_{j=1}^{n-1} (x - \xi_j).$$

Now let O' be any point of $F(D) \cup N_0 \cup N_1$ which is sufficiently close to O with $O' \neq O$. Then,

$$O' = F(x_1, \ldots, x_n)$$

where, by the open mapping theorem applied to a domain of definition of F containing $D \cup M_0 \cup N_0$, x_i is close to ξ_i, $i = 1, \ldots, n-1$, and $x_n \geqslant 0$ with x_n close to 0. The inner product of the normal to N_0 at O with $O - O'$ is

$$\sum_{k=1}^{n} z_k \left\{ \sum_{j=1}^{n-1} (-1)^{j+1} \xi_j^k - \sum_{j=1}^{n} (-1)^{j+1} x_j^k \right\}$$

$$= \sum_{j=1}^{n-1} (-1)^{j+1} \sum_{k=1}^{n} z_k \xi_j^k - \sum_{j=1}^{n} (-1)^{j+1} \sum_{k=1}^{n} z_k x_j^k$$

$$= \sum_{j=1}^{n-1} (-1)^{j+1} Q(\xi_j) - \sum_{j=1}^{n} (-1)^{j+1} Q(x_j)$$

$$= \sum_{j=1}^{n-1} (-1)^{j+1} [Q(\xi_j) - Q(x_j)] + (-1)^n Q(x_n).$$

Now Q attains a local minimum at ξ_1 and alternate maxima and minima at $\xi_2, \ldots, \xi_{n-1}$. It is thus a simple matter to check that the terms

$$(-1)^{j+1} [Q(\xi_j) - Q(x_j)],\ 1 \leqslant j \leqslant n - 1,$$

are nonpositive provided only that x_j is near ξ_j, $1 \leqslant j \leqslant n - 1$, i.e., $\xi_2 \leqslant x_1 < 1$, $\xi_3 \leqslant x_2 \leqslant \xi_1, \ldots, 0 < x_{n-1} \leqslant \xi_{n-2}$. From prior statements, we can select a neighborhood of O to ensure this. Furthermore, $(-1)^n Q(x_n)$ is also always nonpositive when x_n is near zero, i.e., $0 \leqslant x_n \leqslant \xi_{n-1}$, since $Q(0) = 0$. Since $O \neq O'$, some term is not zero and hence the points of $F(D) \cup N_0 \cup N_1$ which are near O lie strictly on one side of H.

If $O = F(1, \xi_2, \ldots, \xi_n) \in N_1$ then Q is defined by

$$Q(0) = 0, \qquad Q'(x) = \prod_{j=2}^{n} (x - \xi_j)$$

and Q achieves a local minimum at ξ_2 and alternate maxima and minima. The inner product in this case is

$$[Q(1) - Q(x_1)] + \sum_{j=2}^{n} (-1)^{j+1} [Q(\xi_j) - Q(x_j)]$$

which is nonnegative if $\xi_2 \leqslant x_1 \leqslant 1$, $\xi_3 \leqslant x_2 < 1,\ldots, 0 < x_n \leqslant \xi_{n-1}$. Thus, if $O' \neq O$ is sufficiently close to O in $F(D) \cup N_0 \cup N_1$, then the inner product is strictly positive.

For the second assertion, suppose that $(\xi_1,\ldots,\xi_{n-1}, 0) \in M_0$ and $F(\xi_1,\ldots,\xi_{n-1}, 0) = O \in L$. Each point R of L' is of the form

$$R = \left\{\tfrac{1}{2}(1-r) + r\sum_{j=1}^{n-1}(-1)^{j+1}\xi_j^k\right\}_{k=1}^{n}, \qquad 0 \leqslant r \leqslant 1.$$

Hence the inner product of the normal to N_0 at O with $O - R$ is

$$\sum_{k=1}^{n} z_k\left\{\sum_{j=1}^{n-1}(-1)^{j+1}\xi_j^k - \tfrac{1}{2}(1-r) - r\sum_{j=1}^{n-1}(-1)^{j+1}\xi_j^k\right\}$$
$$= (1-r)\left\{\sum_{j=1}^{n-1}(-1)^{j+1}Q(\xi_j) - \tfrac{1}{2}Q(1)\right\}.$$

We claim that the term

$$\sum_{j=1}^{n-1}(-1)^{j+1}Q(\xi_j) - \tfrac{1}{2}Q(1)$$

is negative. Note that $\sum_{j=1}^{n-1}(-1)^{j+1}Q(\xi_j)$ is negative and hence the claim surely holds if $Q(1) \geqslant 0$. On the other hand, because $Q'(x) > 0$ for $\xi_1 < x < 1$ we have $Q(1) > Q(\xi_1)$ so that

$$\begin{aligned}\sum_{j=1}^{n-1}(-1)^{j+1}Q(\xi_j) &= Q(\xi_1) - [Q(\xi_2) - Q(\xi_3)] - \cdots\\ &< Q(1) - [Q(\xi_2) - Q(\xi_3)] - \cdots\\ &< Q(1).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{j=1}^{n-1}(-1)^{j+1}Q(\xi_j) - \tfrac{1}{2}Q(1) &= \sum_{j=1}^{n-1}(-1)^{j+1}Q(\xi_j)\\ &\quad - Q(1) + \tfrac{1}{2}Q(1)\\ &< \tfrac{1}{2}Q(1) < 0\end{aligned}$$

when $Q(1) < 0$. This implies that L' lies on the same side of H as $F(D)$.

If L meets N_1 at $O = F(1, \xi_2,\ldots,\xi_n)$ then Q achieves a local minimum at ξ_2 and alternate maxima and minima at $\xi_3,\ldots,\xi_n$ and the inner product of the normal to N_1 at O with $O - R$ is,

$$(1-r)\left\{\sum_{j=2}^{n}(-1)^{j+1}Q(\xi_j) + \tfrac{1}{2}Q(1)\right\}, \tag{2.7}$$

which is nonnegative. Indeed, $(-1)^{n+1} Q(\xi_n)$ is nonnegative since $Q(0) = 0$. Thus, $\sum_{j=2}^{n} (-1)^{j+1} Q(\xi_j)$ is nonnegative. If $Q(1) \geqslant 0$ then (2.7) is surely nonnegative. On the other hand, $-Q(\xi_2) > -Q(1)$ so that (2.7) exceeds $(1 - r)[-(1/2)] Q(1)$, which is nonnegative if $Q(1) < 0$. This concludes the proof of the proposition.

Proof of Theorem 3. Let s_* be a piecewise perfect spline solution of the minimization problem (1.3) which is guaranteed by Theorem 2. In particular, $D^n s_* = \pm\alpha$ on a core subinterval of $[x_1, x_m]$ where $\alpha > 0$ is the extremal constant of the minimization problem. Let $v_1(\epsilon) = \{s_*^{(\nu)}(x_1)\}_0^{n-1}$. If $v_j(\epsilon)$, $1 \leqslant j < j_0 \leqslant m$ have been defined, then by Proposition 1 there exists an n-tuple $v_{j_0}(\epsilon)$, with components uniformly within ϵ of the corresponding components of $\{s_*^{(\nu)}(x_{j_0})\}_0^{n-1}$, such that the equalities

$$\{t^{(\nu)}(x_{j_0-1})\}_0^{n-1} = v_{j_0-1}(\epsilon), \qquad \{t^{(\nu)}(x_{j_0})\}_0^{n-1} = v_{j_0}(\epsilon)$$

fail to hold for every perfect spline t with at most $n - 1$ knots in $[x_{j_0-1}, x_{j_0})$. Let $C(\epsilon)$ consist of all $f \in W^{n,\infty}(x_1, x_m)$ which satisfy the conditions

$$\{f^{(\nu)}(x_i)\}_{\nu=0}^{n-1} = v_i(\epsilon), \qquad i = 1,..., m, \tag{2.8 i}$$

and let

$$\alpha(\epsilon) = \inf\{\| f^{(n)} \|_{L^\infty(x_1,x_m)} : f \in C(\epsilon)\}. \tag{2.8 ii}$$

We claim that

$$\liminf_{\epsilon\to 0} \alpha(\epsilon) \leqslant \alpha. \tag{2.9}$$

Indeed, let H_0 consist of all functions $g \in W^{n,\infty}(x_1, x_m)$ with $g^{(\nu)}(x_1) = 0$, $0 \leqslant \nu \leqslant n - 1$ and consider the mapping L from H_0 into $\mathbb{R}^k$, $k = n(m - 1)$, given by

$$Lg = \{\{g^{(\nu)}(x_i)\}_{\nu=0}^{n-1}\}_{i=2}^{m} .$$

L is clearly surjective. The norm in H_0 can be taken to be

$$\| g \|_{H_0} = \| g^{(n)} \|_{L^\infty(x_1,x_m)}$$

and hence there is a constant c such that each vector in R^k of norm 1 (for convenience, choose the uniform norm) is the image under L of a function in H_0 of norm not exceeding c. It follows that there is an $f_\epsilon \in C(\epsilon)$ with

$$\| f_\epsilon^{(n)} \|_{L^\infty(x_1,x_m)} \leqslant \epsilon c + \alpha$$

from which (2.9) follows.

We now claim that for each $\alpha(\epsilon)$ there is a perfect spline $s(\epsilon)$ satisfying

$$\| s^{(n)}(\epsilon)\|_{L^\infty(x_1,x_m)} = \alpha(\epsilon), \tag{2.10 i}$$

$$\{s^{(\nu)}(\epsilon)(x_i)\}_{\nu=0}^{n-1} = v_i(\epsilon), \qquad i = 1,..., m, \tag{2.10 ii}$$

and the property that $s(\epsilon)$ has at most n knots on each interval (x_i, x_{i+1}), $1 \leqslant i \leqslant n - 1$. Indeed, by Theorem 2, any solution of the extremal problem just considered is a uniquely determined perfect spline on a core subinterval J of $[x_1, x_m]$. The problem of determining $s(\epsilon)$, then, is precisely the problem of extending $s(\epsilon)$ from J to $[x_1, x_m]$. For simplicity, we suppose that $J = [x_1, x_k]$, $2 \leqslant k \leqslant m - 1$. The modifications required for the other possibilities will be apparent. Upon identifying $-1, 0, 1$, respectively, with x_{k-1}, x_k, x_{k+1} via an affine transformation we see that the hypotheses of Proposition 2 are satisfied. Thus we can obtain an extended perfect spline $s(\epsilon)$ on $[x_1, x_{k+1}]$ satisfying

$$\{s^{(\nu)}(\epsilon)(x_i)\}_{\nu=0}^{n-1} = v_i(\epsilon), \qquad i = 1,..., k + 1$$

and

$$D^n s(\epsilon) = \pm\alpha(\epsilon) \text{ on } (x_1, x_{k+1}).$$

The process can clearly be continued to obtain $s(\epsilon)$ satisfying (2.10 i) and (2.10 ii). If the core subinterval is situated differently, and extension must also be carried out from right to left, then it is a routine matter to establish an analog of Proposition 2 treating this case.

Consider the family $\{s^{(n)}(\epsilon) : 0 < \epsilon \leqslant 1\}$.

We extract a subsequence $s^{(n)}(\epsilon_\nu)$, such that $\epsilon_\nu \to 0$, $\alpha(\epsilon_\nu) \to \alpha_* \leqslant \alpha$ and such that the following properties hold.

(i) Each $s(\epsilon_\nu)$ has exactly the same number of knots on $[x_1, x_m)$.
(ii) For each i, the sequence $\xi_i^{(\nu)}$ of ith knots of $s(\epsilon_\nu)$ is convergent.
(iii) For each i, the sequence of numbers

$$s^{(n)}(\epsilon_\nu)|_{(\xi_i^{(\nu)}\xi_{i+1}^{(\nu)})} \text{ is convergent.}$$

A simple consequence of (i), (ii) and (iii) is that $s^{(n)}(\epsilon_\nu)$ converges in $L^2(x_1, x_m)$ to a limit function s_n which is a step function with values $\pm\alpha_*$ and discontinuities restricted to the points ξ_i described in (ii),

It is now an easy matter to verify that the family $\{s^{(n-1)}(\epsilon_\nu)\}$ is a uniformly bounded and equicontinuous family on $[x_1, x_m]$. Indeed,

$$s^{(n-1)}(\epsilon_\nu)(x) = s_*^{(n-1)}(x_1) + \int_{x_1}^{x} s^{(n)}(\epsilon_\nu)(u)\, du$$

so that

$$\| s^{(n-1)}(\epsilon_\nu)\|_{L^\infty(x_1,x_m)} \leqslant | s_*^{(n-1)}(x_1)| + C(x_m - x_1), \qquad \nu \geqslant 1,$$

and

$$| s^{(n-1)}(\epsilon_\nu)(x) - s^{(n-1)}(\epsilon_\nu)(y)| = \left| \int_x^y s^{(n)}(\epsilon_\nu)(u)\, du \right| \leqslant C\, | y - x |$$

for each $\nu \geqslant 1$. Thus, the Arzela-Ascoli theorem yields a uniformly convergent subsequence of $s^{(n-1)}(\epsilon_\nu)$, say, $s^{(n-1)}(\epsilon_{\nu_k})$. Elementary considerations show that, actually, $s^{(\mu)}(\epsilon_{\nu_k})$ is uniformly convergent on $[x_1, x_m]$ to a continuous function s_μ, $0 \leqslant \mu \leqslant n-1$.

Standard arguments now show that $s_0 \in W^{n,\infty}(x_1, x_m)$ with

$$s_0^{(\mu)} = s_\mu, \qquad 0 \leqslant \mu \leqslant n,$$

and s_0 is a perfect spline with at most n knots on (x_j, x_{j+1}), $1 \leqslant j \leqslant m-1$, satisfying $s_0^{(n)} = \pm\alpha_*$. However, the uniform convergence of $s^{(\mu)}(\epsilon_\nu)$, $0 \leqslant \mu \leqslant n-1$, and the property that $\epsilon_\nu \to 0$ imply

$$s_0^{(\mu)}(x_i) = s_*^{(\mu)}(x_i), \qquad 0 \leqslant \mu \leqslant n-1, \qquad 1 \leqslant i \leqslant m. \tag{2.11}$$

It follows immediately that $\alpha_* = \alpha$ and the proof is completed since (2.11) implies, in particular, that s_0 interpolates the appropriate constrained values at the nodes, inasmuch as s_* does.

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